Quantizing Noise and Data Transmission

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Methods for calculating the power in the quantizing noise on digital transmission facilities have been known for some time. A more difficult but unavoidable problem is the effect that this noise has on data signals intended for analog transmission. This paper demonstrates that to assume that the noise will behave as a white Gaussian noise process will always (except for a simple factor) yield an upper bound on the probability of error when no companding is present. We assume that linear detection will be used, as for a PAM system, and the result is true whether or not filtering or demodulation is involved. Results are illustrated by applying them to a model of an existing VSB modem whereby the additional degradations resulting from data set imperfections are included as added baseband noise.

A modem operating perfectly would make no errors at all at the higher transmission levels. For example, with no companding, a set with an eight-level eye closed by even 30 percent would not yield errors for input powers down to −15 dBm. Thus quantizing noise is not a basic limiting factor in the error rate for all input levels. A similar rigorous theory is not available for compandored systems, but for special situations reasonable estimates can be made. For logarithmic companding and eight-level VSB transmission, worst case estimates indicate error rates about 10^{-6} *for one link of T1 carrier.*

I. INTRODUCTION AND SUMMARY

The T1 carrier system is a digital transmission scheme for analog signals.[1] Even though the digits in the coded bit stream might be transmitted without error, when the analog signal (which may in fact be a data signal designed for analog facilities) is reconstructed at the receiving terminal, quantizing noise is inevitably added and can be large enough to cause errors in the customer's data.

We show that, under some simple constraints between sampling rates and bandwidths which are satisfied in practice, and independent of the particular data signal used, an upper bound on error rates is obtained if the quantizing noise is assumed to be a white Gaussian process of power $\Delta^2/12$ and bandwidth $1/2T_1$.* We assume that linear detection will be used, and the result is true even if additional filtering is done (as one might do with a receiving filter). And it is true whether or not a demodulation process takes place. Using the model in Fig. 1 for the digital transmission scheme, results are specialized to obtain error rates for eight-level VSB transmission (Fig. 2). Imperfections of the data set are included as added baseband noise. If it were not for these imperfections, error free transmission would result over an appreciable range of power levels (see Table I). For a logarithmic compandor and VSB data, even using worst case estimates, the error rate for one link is quite low, about 10^{-6}.

II. QUANTIZED TRANSMISSION SCHEME

Let us consider a transmission scheme for a single channel that, for our purposes, typifies the T1 carrier system. As suggested in Fig. 1, the signal to be transmitted is assumed not to have any power beyond B Hz. The signal is sampled at the Nyquist rate $T_1 = 1/(2B)$ and these samples are passed through an instantaneous nonlinear device with characteristic $v_{\text{out}} = F(v_{\text{in}})$. The compressed samples are then quantized by a uniform quantizer of step size Δ, and coded into binary sequences. The binary sequences are assumed to be transmitted without error and the process is reversed: sequences are decoded into pulses, expanded according to the inverse function $F^{-1}(x)$ and the impulses are used to excite an ideal filter of bandwidth B and amplitude T_1.† A receiving filter G generally follows the ideal filter and we include this in our description, although it would not be part of a T1 transmission system. If the bandwidth of G is entirely contained in B then one may consider the impulses to excite T_1G directly.

To be more specific, we are concerned with two particular compandor characteristics $F(x)$. One is $F(x) = x$, that is, quantizing

* Here Δ is the quantizer step size and $1/T_1$ is the sampling rate. Also this statement is true only modulo a simple factor given in the text.

† The amplitude gain of the ideal output filter for the carrier system is chosen to be T_1 in order that the signal component will undergo no gain relative to its sampled values at the transmitter.

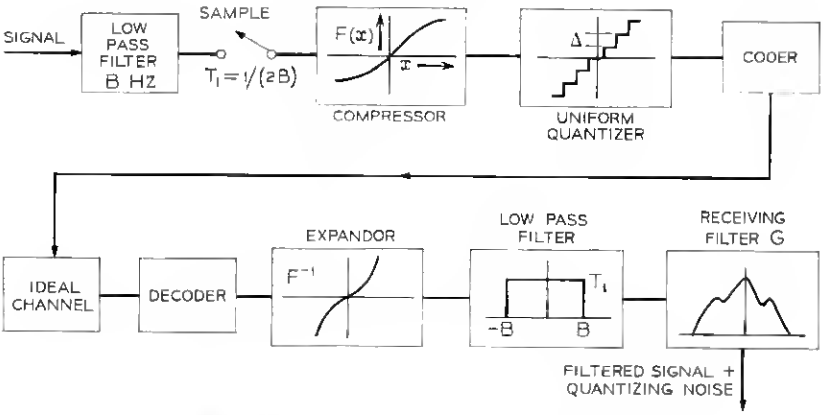

Fig. 1 — Essential elements of quantized transmission scheme.

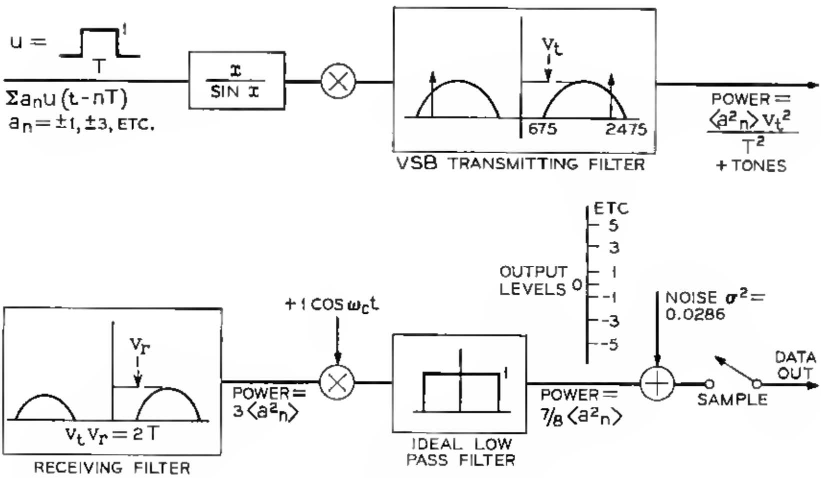

Fig. 2 — Model of back-to-back transmission of VSB modem. The noise added at baseband represents imperfections of the modem.

TABLE I—QUANTIZING NOISE*

Mean square P(dBm)	n_{max}
−5	0.221
−10	0.394
−15	0.698
−20	1.245
−25	2.21
−30	3.94
−35	6.98
−40	12.45
−45	22.1

* Peak values of quantizing noise as a function of input power P. The noise scale is such that a perfect receiver would make no errors for $n_{max} < 1$. The power scale is such that the quantizer overloads at an instantaneous power of +6 dBm.

without compandoring. The other case is (in normalized units)

$$
\begin{aligned}
F(x) &= -F(-x) \\
F(x) &= \frac{\ln(1+\mu x)}{\ln(1+\mu)}, \qquad 0 \leqq x \leqq 1 \\
&= 1 \qquad\qquad\qquad\quad x > 1,
\end{aligned}
\tag{1}
$$

where μ, the degree of compandoring, is large. Typically, $\mu = 100$ for a good approximation to existing devices.

Finally, when specific values are required, we assume 7 bit coding to be used for the quantized samples and use $\Delta = 1/63$.

We hasten to add that quantizing noise is not the only degrading factor for the existing T_1 facilities. Apparently mismatch and mistracking of compressor and expandor cause nonlinearities which are responsible for peculiar behaviors of error rate versus signal power curves.[2]

III. GENERAL THEORY

Let us represent the signal $l(t)$ which is to be sampled and quantized by

$$l(t) = x(t)\cos\omega_c t - y(t)\sin\omega_c t, \tag{2}$$

and the sampling wave as

$$\sum_k \delta(t - kT_1 - \tau), \tag{3}$$

where the random timing phase is uniformly distributed over the interval $0 \leqq \tau \leqq T_1$. The pulse trains representing (2) immediately after sampling, compression, and quantization are given by expressions (4), (5), and (6), respectively.

$$\Sigma l(kT_1 + \tau)\delta(t - kT_1 - \tau); \tag{4}$$

$$\Sigma l_{\text{comp}}(kT_1 + \tau)\delta(t - kT_1 - \tau); \tag{5}$$

$$\Sigma \hat{l}_{\text{comp}}(kT_1 + \tau)\delta(t - kT_1 - \tau), \tag{6}$$

where

$$l_{\text{comp}}(kT_1 + \tau) = F[l(kT_1 + \tau)]$$

is the compressed sample value and $\hat{l}_{\text{comp}}(t)$ is the particular one of the $(2^N - 1)$ levels that the quantizer output gives as the value for $l_{\text{comp}}(t)$. If we let the subscript "exp" stand for the result of operation of the expandor at the receiving terminal, then the impulse associated with time $(kT_1 + \tau)$ has area

$$[\hat{l}_{\text{comp}}(kT_1 + \tau)]_{\text{exp}} = l(kT_1 + \tau) + \epsilon(kT_1 + \tau). \tag{7}$$

Because the expandor has as its input an estimate of the compressed pulse area, the error term $\epsilon(kT_1 + \tau)$ is not zero but may take any value in an interval, that is,

$$\epsilon(t) \,\varepsilon \left[\frac{-\Delta(t)}{2}, \frac{\Delta(t)}{2}\right]. \tag{8}$$

The spread $\Delta(t)$ that the quantizing error may take is not necessarily equal to the quantizer step size Δ when companding is present, but is given by the formula (see Appendix A)

$$\Delta(t) = \frac{\Delta}{|F'[l(t)]|}. \tag{9}$$

In (9), $F'[l(t)]$ is the derivative of the compressor characteristic evaluated at that input amplitude of the signal at the time of the sampling. The error signal generated at the output of the receiving filter is obtained by convolving the impulse train

$$\Sigma\epsilon(kT_1 + \tau)\delta(t - kT_1 - \tau) \tag{10}$$

with the impulse response $T_1 g(t)$ of the receiving filter.* Denoting

* Again, $g(t)$ is associated with the receiving filter of the data set and the constant T_1 is the gain of the ideal output filter of the carrier system.

this noise by $n_l(t)$ we have

$$n_l(t) = T_1 \Sigma \epsilon(kT_1 + \tau) g(t - kT_1 - \tau). \tag{11}$$

To proceed further we make the assumption tbat the quantizing error $\epsilon(t)$ of the output sample in (10) is uniformly distributed over the interval

$$\left[\frac{-\Delta(t)}{2}, \frac{\Delta(t)}{2}\right],$$

thus having mean zero and variance $\Delta^2(t)/12$, and that different sample errors are independent. Notice that the latter assumption is not the same as assuming that different sample *values* are independent.

IV. SUMS OF UNIFORM VARIATES

As (11) illustrates, a basic problem which must be dealt with is the probability distribution of sums of independent and uniformly distributed random variables. We will obtain an upper bound on the tail probabilities of interest by applying the technique of tbe Chernoff bound.[3, 4] This bounding technique states that if a probability density $p(x)$ has a moment generating function (*mgf*) $M(s)$, where

$$M(s) = \int_{-\infty}^{\infty} [\exp(sx)] p(x)\, dx, \tag{12}$$

then

$$Q = \text{Prob}\,[x \geqq a] \leqq M(s) \exp(-sa), \qquad s \geqq 0. \tag{13}$$

Thus to obtain an upper bound one simply multiplies the moment generating function by an exponential, both evaluated at an arbitrary positive s. Actually it is known that there is a best s to choose, and it is that one, if it exists, which satisfies the equation

$$\frac{d}{ds} \ln M(s) = a. \tag{14}$$

Equation (14) assures a stationary value for the right side of (13) and it can be shown tbat sucb an s in fact minimizes $M(s)e^{-sa}$.

For example, for a Gaussian variate of mean m and variance σ^2, the moment generating function is well known to be given by

$$M(s) = \exp\left[ms + \frac{s^2\sigma^2}{2}\right]. \tag{15}$$

Thus the best s to choose is, using (14),

$$s = \frac{(a - m)}{\sigma^2}. \tag{16}$$

Notice that only if $a \geqq m$ is this $s \geqq 0$. Thus, as long as a is greater than the mean, we have for the Gaussian case

$$Q \leqq \exp\left[-\frac{(a - m)^2}{2\sigma^2}\right], \tag{17}$$

where (17) results from using (15) and (16) in (13). For the Gaussian variate under discussion the exact answer is also well known to be given by

$$Q = \tfrac{1}{2} \operatorname{erfc} \frac{(a - m)}{(2)^{\frac{1}{2}}\sigma}, \tag{18}$$

where erfc x is the coerror function.[5] In addition, equation (7.1.13) of Ref. 5 states that

$$\exp(-x^2) \leqq (\pi)^{\frac{1}{2}}[x + (x^2 + 2)^{\frac{1}{2}}][\tfrac{1}{2} \operatorname{erfc} x], \tag{19}$$

and hence the difference between the Chernoff answer (17) and the exact answer (18) for the Gaussian case is no more than the multiplicative factor $(\pi)^{\frac{1}{2}}[(\rho)^{\frac{1}{2}} + (\rho + 2)^{\frac{1}{2}}]$ where $(\rho)^{\frac{1}{2}} = (a - m)/[(2)^{\frac{1}{2}}\sigma]$.

We modify this procedure for our problem with the following obvious lemma.

Lemma 1: *If $G(s)$ is an upper bound for the moment generating function, that is, $M(s) \leqq G(s)$ for all s, then*

$$Q \leqq e^{-sa}G(s), \qquad s \geqq 0. \tag{20}$$

In particular, a positive $s = s_0$ which satisfies

$$\frac{d}{ds} \ln G(s) \bigg|_{s=s_0} = a \tag{21}$$

is legitimate.

Next consider a random variable x which is uniformly distributed over $[-\Delta/2, \Delta/2]$. The variance of this variable is $\Delta^2/12$, and it has a moment generating function $M(s)_{\text{unif}}$

$$M(s)_{\text{unif}} = \frac{\sinh \frac{s\Delta}{2}}{\frac{s\Delta}{2}} = \sum_{n=0}^{\infty} \left(\frac{s\Delta}{2}\right)^{2n} \frac{1}{(2n + 1)!}. \tag{22}$$

Now the nth term of the sum in (22) is positive and upper bounded by

$$\left[\left(\frac{s\Delta}{2}\right)^2 \frac{1}{6}\right]^n \frac{1}{n!}.$$

Hence

$$M(s)_{\text{unif}} \leqq \sum_{n=0}^{\infty} \left[\left(\frac{s\Delta}{2}\right)^2 \frac{1}{6}\right]^n \frac{1}{n!} = \exp\left[\frac{\Delta^2}{12}\frac{s^2}{2}\right]. \tag{23}$$

Tbus we have shown that the moment generating function of a zero-mean uniform density is upper bounded by tbat of a zero-mean Gaussian having the same variance.* If the uniform variable has mean m the theorem is still true if we use instead the moment generating function of a Gaussian witb mean m.

We are now ready to write down a whole class of random variables which have moment generating functions upper bounded by those of a Gaussian of the same variance. Suppose the result is true for two independent random variables, x and y, of variances σ_x^2 and σ_y^2, namely

$$\begin{aligned} M_x(s) &\leqq \exp\left[\frac{s^2\sigma_x^2}{2}\right] \\ M_y(s) &\leqq \exp\left[\frac{s^2\sigma_y^2}{2}\right]. \end{aligned} \tag{24}$$

Then using the theorem that the moment generating function of a sum of two independent random variables is the product of their individual moment generating functions, we have

$$\begin{aligned} M_{x+y}(s) &= M_x(s)M_y(s) \leqq \exp\left[\frac{s\sigma_x^2}{2}\right]\exp\left[\frac{s\sigma_y^2}{2}\right] \\ &= \exp\left[\frac{s^2}{2}(\sigma_x^2 + \sigma_y^2)\right] = \exp\left[\frac{s^2\sigma_{x+y}^2}{2}\right], \end{aligned}$$

where

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$$

is the variance of $(x + y)$. Thus the moment generating function of a sum of any number of independent uniforms of arbitrary means and variances is upper bounded by the appropriate Gaussian one (same mean and variance as the sum), and thus use of (17) through (21) provides a rigorous upper bound for tail probabilities of the sum.

* A similar theorem was discussed by Saltzberg for the case of equally spaced delta functions.[6] We have followed his method of proof here.

V. FURTHER ANALYSIS WITHOUT COMPANDING

When no companding is present, tbe independent random variables $\epsilon(kT_1 + \tau)$ have variance $\Delta^2/12$ and the variance of the noise (11) is

$$\sigma_i^2 \equiv \langle n_i^2(t) \rangle = T_1^2 \cdot \frac{\Delta^2}{12} \cdot \sum_{k=-\infty}^{\infty} g^2(t - kT_1 - \tau). \tag{25}$$

We evaluate the infinite sum in (25) by using the Poisson sum formula, namely

$$\sum_{k=-\infty}^{\infty} g^2(t - kT_1 - \tau) = \frac{1}{T_1} \sum_{m=-\infty}^{\infty} \exp\left[\frac{2\pi i m(t-\tau)}{T_1}\right] G_2\left(m \frac{2\pi}{T_1}\right), \tag{26}$$

where $G_2(\omega)$ is the Fourier transform of $g^2(t)$. Now since the bandwidth of the filter G is assumed not to exceed $1/2T_1$ Hz, only the $m = 0$ term of (26) contributes and we obtain

$$\sigma_i^2 = T_1 \cdot \frac{\Delta^2}{12} \cdot G_2(0). \tag{27}$$

Equation (27) implies that the noise power measured before the receiving filter is $\Delta^2/12$. This result has been obtained by Bennett[7] who also showed that the spectrum of this noise is flat across the band. Further, equation (27) is consistent with filtering white noise since

$$G_2(0) = \int_{-\infty}^{\infty} g^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2\, d\omega. \tag{28}$$

An important fact about (27) is that the received passband noise power without companding is independent of many properties of the signal. Thus it is independent of signal power and multilevel structure. It is not independent of rate, however, since this enters implicitly into the factor $G_2(0)$, and likewise it is not independent of roll-off. By halving the speed and doubling the number of levels, one decreases the noise by 3 dB, but loses 6 dB in noise margin, thus leaving one with a net loss of 3 dB in noise margin. Thus it is best to use as few levels as possible consistent with given speed objectives, at least if the quantizing noise behaves anything like Gaussian noise.

Let us discuss further some statistical aspects of the quantizing noise at baseband. The "line" signal must be demodulated as in VSB transmission by multipling the (filtered) received signal by cos $\omega_c t$ and eliminating double frequency components. We represent the impulse response $g(t)$ of the passband receiving filter G by

$$g(t) = g_x(t) \cos \omega_c t - g_y(t) \sin \omega_c t. \tag{29}$$

Further we specialize to the practical constraints* $\omega_c > B_i$ and $2\pi/T_1 > 2(\omega_c + B_i)$. The demodulated noise is

$$n_b(t) = \frac{T_1}{2} \Sigma\epsilon(kT_1 + \tau)\{g_x(t - kT_1 - \tau) \cos(\omega_c kT_1 + \omega_c\tau) + g_y(t - kT_1 - \tau) \sin[\omega_c kT_1 + \omega_c\tau]\}. \tag{30}$$

The general expression (29) can be simplified for a VSB receiving filter which is symmetric about midband frequency ω_1, and linear phase characteristic, by writing

$$\begin{aligned} g(t) &= g_1(t) \cos \omega_1 t \\ \omega_c - \omega_1 &= \pi/2T, \end{aligned} \tag{31}$$

where $1/T$ is the symbol repetition frequency. Of course a g_x and a g_y may be immediately written down from (31).

From (30) we derive in Appendix B, equation (32) for the baseband variance $\sigma_b^2(t)$:

$$\sigma_b^2(t) = \frac{1}{2}\left(\frac{1}{4}\frac{\Delta^2}{12}\right)T_1[G_{2x}(0) + G_{2y}(0)], \tag{32}$$

where $G_{2i}(\omega)$ is the Fourier transform of $g_i^2(t)$. We now will show that this result is identical to the baseband noise power that would appear if flat Gaussian noise of power $\Delta^2/12$ were on the line. We do not regard this as obvious; in fact it is not true that the *signal* power at baseband is the same as if one had Gaussian noise of the same power and spectrum on the line that the signal bas. The proof depends on a few simple observations. If passband Gaussian noise is represented by

$$n(t) = n_x(t) \cos \omega_c t - n_y(t) \sin \omega_c t, \tag{33}$$

then

$$\sigma_n^2 \equiv \langle n^2 \rangle = \langle n_x^2 \rangle = \langle n_y^2 \rangle, \tag{34}$$

and so baseband noise power is $\sigma_n^2/4$. Next we notice that white Gaussian noise, having same total power as quantizing noise over the band $(-1/2T_1, 1/2T_1)$ Hz, has two sided spectral density

$$N(\omega) = N_0/2 = \frac{\Delta^2}{12}\cdot T_1 \text{ watts per cycle.} \tag{35}$$

* B_i is the bandwidth of $g_i(t)$, $i = x, y$.

Thus the Gaussian noise power out of the receiving filter G would be

$$\begin{aligned}\sigma_n^2 &= \frac{\Delta^2}{12}\cdot T_1 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mid G(\omega) \mid^2 \\ &= \frac{\Delta^2}{12} T_1 \int_{-\infty}^{\infty} g(t)^2 \, dt \\ &= \frac{\Delta^2}{12} T_1 \left[\int_{-\infty}^{\infty} \frac{g_x^2(t) + g_y^2(t)}{2} \, dt + \int_{-\infty}^{\infty} \frac{g_x^2(t) - g_x^2(t)}{2} \cos 2\omega_c t \, dt \right. \\ &\qquad \left. + \int_{-\infty}^{\infty} g_x(t) g_y(t) \sin 2\omega_c t \, dt \right]. \end{aligned} \tag{36}$$

Since neither $g_x(t)$ nor $g_y(t)$ are assumed to have any frequencies as high as ω_c the last two integrals above vanish. The final remark that completes the proof is

$$\int_{-\infty}^{\infty} g_i^2(t) \, dt = G_{2i}(0).$$

Thus noncompandored quantizing noise behaves, at least concerning, its power, as zero mean white Gaussian noise, flat over the band $(-1/2T_1, 1/2T_1)$ Hz, and total power $\Delta^2/12$. This statement is true with or without demodulation.

One would like to go further and treat the baseband noise as zero mean Gaussian of variance given by (32). There is a justification for making this additional step. Recall the result of Section IV, which stated that if

$$\begin{aligned} z &= \Sigma \Delta_i \\ \langle z \rangle &= \Sigma \langle \Delta_i \rangle \\ \sigma^2 &= \operatorname{var} z = \tfrac{1}{12} \Sigma \Delta_i^2 \end{aligned} \tag{37}$$

is a sum of independent and uniformly distributed variates Δ, , then (provided $\sigma^2 = 1/12 \Sigma \Delta_i^2 < \infty$) for all A such that $A > \Sigma \langle \Delta_i \rangle$,

$$\text{Prob } (z > A) \leqq (\pi)^{\frac{1}{2}} [(\rho)^{\frac{1}{2}} + (\rho + 2)^{\frac{1}{2}}] P_g(A). \tag{38}$$

In (38), $(\rho)^{1/2} = A/[(2)^{1/2}\sigma]$, and $P_g(A)$ is the probability that a Gaussian variate of the same mean and variance as z is greater than A. Since $P_g(A)$ depends exponentially on ρ, the coefficient structure in (38) is not nearly as important as $P_g(A)$. We would like to argue (but not prove) that ignoring the coefficient in (38), that is, simply

assuming Gaussian behavior, is quite accurate for the baseband noise (30) for the error rates of practical interest.

Thus consider eight-level, 50 percent roll off, transmission over our hypothetical "noncompandored T_1" transmission facility. From (30) and (31)

$$n_b(t) = \frac{T_1}{2} \sum_k \epsilon(kT_1 + \tau)h_1(t - kT_1 - \tau) \cdot \cos [(\omega_c - \omega_1)(t - kT_1 - \tau) + \omega_c(kT_1 + \tau)], \tag{39}$$

where, according to an appropriate normalization,

$$h_1(t) = \frac{4v_r}{\pi T} \frac{\cos \frac{\pi t}{T}}{1 - \left(\frac{2t}{T}\right)^2}. \tag{40}$$

Notice that since $h_1 \approx 1/t^2$, t large, the sum in (39) is bounded. A computer study of (39) for various values of t and τ shows this bound to be not too sensitive (about 5 percent variations) to choices of t and τ. Numerically we find

$$| n_b(t) | \leqq \frac{T_1}{2} \cdot \frac{\Delta}{2} \cdot \left(\frac{4v_r}{\pi T}\right)(5.31). \tag{41}$$

From the sum formula (26) the variance of (39) is obtained. We calculate

$$\sigma_b^2 = \frac{1}{4} \frac{\Delta^2}{12} \frac{T_1 v_r^2}{T}. \tag{42}$$

Thus a peak-to-mean square ratio of the baseband noise power may easily be shown to be 15 dB. To obtain some insight from this value, consider the question of how many (N) identical independent, zero-mean uniform densities one would have to convolve to get a peak to rms value of 15 dB; the answer is $N = 10$. Ten uniforms generate, we feel, a reasonable approximation to a Gaussian curve. As a check, consider that our ten uniform densities each have range $[-0.5, 0.5]$. To check (not prove) tbe approximation on the tails we calculate

$$\text{Prob [sum} \geqq 4.5] = \frac{1}{10!} = 2.75 \times 10^{-7}.$$

The Gaussian assumption gives 4.46×10^{-7}. Thus we will assume that for error rates $> 10^{-7}$ the Gaussian assumption will yield reasonably accurate answers, not just being a bound in the sense discussed above.

The above theory showing that noncompandored quantizing noise may be considered to he additive white Gaussian noise with zero mean, variance $\Delta^2/12$, of 4 kHz handwidth has heen compared with the experimental results of Gustafson on the performance of the VSB (203) data set which operates at 5400 hits per second. Fortunately an experimental curve is available for error rate versus signal-to-noise ratio without companding and this is shown in Fig. 3 along

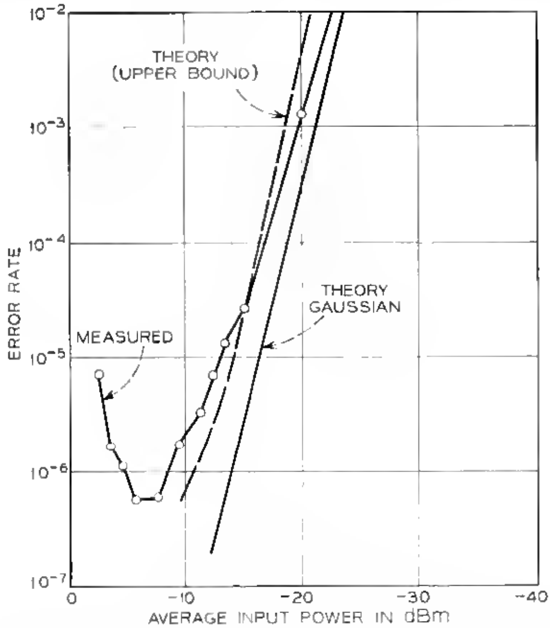

Fig. 3 — Comparison of experimental and theoretical error rates for one link of noncompandored transmission. Theory neglects overload distortion. Instantaneous input power of +6 dBm is the onset of overload.

with the results of present theory* (for one link of T1.) The rise in the experimental curve at high input signal power results from overload distortion of the quantizer which has heen neglected for the present analysis. Overload occurs at a peak power of +6 dBm on the scale used in Fig. 3, and thus the peak power to average power for the eight-level VSB set (including pilot tones) appears to he around 11 dB. In general the ohserved error rate is higher than the theoret-

* To model the performance of the actual 203 receiver, an additional noise source is included at baseband, as suggested by Saltzberg[8] and shown in Fig. 2. The baseband S/N for this noise is chosen to be 28.08 dB. This noise alone would yield an error rate of 2.5×10^{-9}.

ical prediction, and for low P_e is even higher than the theoretical upper bound. Perhaps this is caused by other distortions in the system not considered here.

For multiple links of transmission one should take the quantizing noise to have the same properties as above, but the total noise power is $N \cdot \Delta^2/12$, where N is the number of links.

We wish to emphasize that the curve drawn in Fig. 3 does not represent any theoretical ideal; we have tried to understand the performance of an existing data set and its imperfections. Actually, if the data set were functioning perfectly, there is a range of input power where no errors would be made. We have normalized units so that no errors can be made if the baseband noise is less than unity in magnitude. Table I shows the peak value of quantizing noise calculated from equation (41) as a function of input power measured in the same units as in Fig. 3. An input power of -15 dB would be near typical operating levels. If the data set were imperfect but the eight-level eye were no more than 30 percent closed (but one had perfect timing), then one would still not make errors down to -15 dBm. In general we see that quantizing noise is not a *basic* limiting factor on the error rate for all input power levels.

VI. ANALYSIS WITH COMPANDING

Equation (9) indicates that the derivative of the compressor characteristic is an important quantity. For the logarithmic curve given in equation (1),

$$F'(x) = \frac{\mu}{\ln(1+\mu)} \frac{1}{1+\mu|x|}. \tag{43}$$

The average of $\Delta^2(t)$ now is not $\Delta^2/12$ but is

$$\begin{aligned}\sigma_{\text{av}}^2 = \langle \Delta^2(t) \rangle &= \frac{\Delta^2}{12} \cdot \left\langle \frac{1}{|F'(x)|^2} \right\rangle \\ &= \frac{\Delta^2}{12} \left[\frac{\ln(1+\mu)}{\mu} \right]^2 (1 + 2\mu\langle |x| \rangle + \mu^2 P),\end{aligned} \tag{44}$$

where the average power $P = \langle x^2 \rangle$. Now $\langle |x| \rangle$ cannot be less than zero nor more than $(P)^{1/2}$. Hence

$$\kappa(1 + \mu^2 P) \leqq \sigma_{\text{av}}^2 \leqq \kappa[1 + \mu(P)^{\frac{1}{2}}]^2 \tag{45}$$

where

$$\kappa = \frac{\Delta^2}{12} \left[\frac{\ln(1+\mu)}{\mu} \right]^2. \tag{46}$$

The lower and upper bounds in (45) indicate that for large $\mu(P)^{\frac{1}{2}}$ the average noise power is not a sensitive function of the probability density of the input signal. The knowledge of σ_{av}^2 cannot be used here to obtain a strict upper bound for the probability of error as was done in tbe uncompandored case, for tbe "instantaneous" noise variance is correlated witb signal values. Thus large input signals "see" bigger step sizes, in effect, than smaller inputs would. One concludes from this that for multilevel transmission the outermost levels would have tbe greatest noise associated with them.

To make exact calculations on this matter is a difficult task, and we confine ourselves to some estimates of tbe effects. Estimates can be obtained by restricting attention to special sequences. Thus for an eight-level PAM system let an arbitrary sequence consisting only of the outer levels (± 7) be transmitted, and compare this witb another sequence consisting of (± 5) transmitted in place of (± 7). Then the quantizing noise will be—considering the μ^2 term in (44) to be of principal importance—in the ratio $7^2/5^2$. Thus tbe outer level will have, in this circumstance, 3 dB more noise tban the next inner level. The contrast between these levels will be somewbat lessened in a random sequence using all levels, but it is clear tbat the 3 dB number quoted here provides an upper bound to the difference.

Worst case estimates of error rate in the compandored case may be made by replacing $\Delta^2/12$ in (42) by tbe upper bound for σ_{av}^2 given in (45), and finally using peak power instead of average power in (45).

For the eigbt-level VSB system considered previously, operating on $T1$ facilities, tbis procedure yields error rates of $10^{-5} - 10^{-6}$ over one transmission link (for the interesting ranges of input power).

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APPENDIX A

Derivation of Quantization Error

Equation (9) of the text relating the output sample error $\Delta(t)$ to step size Δ, compressor characteristic $F(x)$, and signal amplitude $l(t)$ at time of sampling is easy to derive if the chain rule is used to differentiate the relation

$$F^{-1}[F(x)] = x \tag{47}$$

to obtain

$$\left.\frac{dF^{-1}_{(u)}}{du}\right|_{u=F(x)} \left.\frac{dF(v)}{dv}\right|_{v=x} = 1. \tag{48}$$

Now clearly, if the error made when l_{comp} is quantized is small, that is, if Δ is small, then

$$\begin{matrix}\text{output}\\ \text{sample}\end{matrix} = F^{-1}(l_{\text{comp}}) + \Delta \left.\frac{dF^{-1}}{du}\right|_{u=l_{\text{comp}}}, \tag{49}$$

or

$$\begin{matrix}\text{output}\\ \text{sample}\end{matrix} = l + \frac{\Delta}{F'(l)}, \tag{50}$$

where (48) has been applied to (49) to obtain (50).

APPENDIX B

Derivation of σ_b

Squaring (30) and averaging over $\{\epsilon\}$ gives

$$\begin{aligned}\sigma_b^2(t) = \frac{T_1^2}{4}\cdot\frac{\Delta^2}{12}\cdot \sum \{&g_x^2(t - kT_1 - \tau)\\ &\cdot \cos^2 [\omega_c kT_1 + \omega_c \tau] + g_v^2(t - kT_1 - \tau)\\ &\cdot \sin^2 [\omega_c kT_1 + \omega_c\tau] + 2g_x(t - kT_1 - \tau)g_v(t - kT_1 - \tau)\\ &\cdot \sin [\omega_c kT_1 + \omega_c\tau)] \cos [\omega_c kT_1 + \omega_c \tau]\}\end{aligned} \tag{51}$$

or,

$$\begin{aligned}\sigma_b^2(t) = \frac{1}{2}\left(\frac{T_1^2}{4}\frac{\Delta^2}{12}\right) &\sum [g_x^2 + g_v^2]\\ &+ \frac{1}{2}\left(\frac{T_1^2}{4}\frac{\Delta^2}{12}\right)\{\sum (g_x^2 - g_v^2) \cos [2\omega_c kT_1 + 2\omega_c\tau]\\ &+ 2 \sum g_x g_v \sin [2\omega_c kT_1 + 2\omega_c \tau]\}.\end{aligned} \tag{52}$$

All the sums in (52) may be evaluated using the Poisson sum formula quoted in equation (26). The first term on the right of (52) is simplest to handle. Since g_i has no frequencies higher than $1/2T_1$, the Fourier transform $G_{2i}(\omega)$ of $g^2(t)$ has support contained in $[-2\pi/T_1, 2\pi/T_1]$, and further, since it is a convolution, $G_{2i}(\pm 2\pi/T_1) = 0$. Thus

$$\frac{1}{2}\left(\frac{T_1^2}{4}\frac{\Delta^2}{12}\right) \sum (g_x^2 + g_v^2) = \frac{1}{2}\left(\frac{1}{4}\frac{\Delta^2}{12}\right)T_1[G_{2x}(0) + G_{2v}(0)]. \tag{53}$$

The other sums in (52) are all zero, for a typical sum is (where $\theta = 2\omega_c T_1$)

$$\sum f(t - nT - \tau) \cos [n\theta + \varphi]$$
$$\equiv \cos \left[t \frac{\theta}{T_1} + \varphi - \frac{\tau\theta}{T_1} \right] \sum f(t - nT - \tau) \cos (t - nT_1 - \tau) \frac{\theta}{T_1}$$
$$+ \sin \left[t \frac{\theta}{T_1} + \varphi - \frac{\theta}{T_1} \right] \sum f(t - nT - \tau) \sin (t - nT_1 - \tau) \frac{\theta}{T_1} \cdot \tag{54}$$

The sum formula is now directly applicable to the functions $f(t) \cos 2\omega_c t$ and $f(t) \sin 2\omega_c t$. The functions have Fourier transforms which, according to the discussion following equation (29), vanish at $\omega = \pm 2\pi/T_1 \cdot k$, where k is any integer, *including* zero. The results claimed follow.

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